Aperiodic order, integrated density of states and the continuous algebras of John von Neumann

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Abstract. In [5] Lenz and Stollmann proved the existence of the integrated density of states in the sense of uniform convergence of the distributions for certain operators with aperiodic order. The goal of this paper is to establish a relation between aperiodic order, uniform spectral convergence and the continuous algebras invented by John von Neumann. We illustrate the technique by proving the uniform spectral convergence for random Schrödinger operators on lattices with finite site probabilities, percolation Hamiltonians and for the pattern-invariant operators of self-similar graphs.

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1 Introduction

First, let us recall the notion of uniform spectral convergence studied by Lenz and Stollmann [4], [5]. Let G be a simple, connected infinite graph with bounded vertex degrees, with vertex set V and edge set E. Let $\Delta_G : L^2(V) \to L^2(V)$ be the combinatorial Laplacian operator:

$$\Delta_G f(x) = \deg(x) f(x) - \sum_{(x,y)\in E} f(y)\,.$$

A Følner-sequence in G is a sequence of finite subsets $\{Q_n\}_{n=1}^\infty \subset V$ such that $\frac{|\partial Q_n|}{|Q_n|} \to 0$, where

$$\partial Q_n := \{p \in Q_n \mid \text{there exists}\, q \notin Q_n, (p,q) \in E\}\,.$$

For a subset $A \subset V$, let $\Delta_A : L^2(A) \to L^2(A)$ be the Laplacian on the subgraph G_A spanned by the vertices of A. If A is finite, then it is a positive, self-adjoint linear transformation. The normalized spectral distribution function of Δ_A is defined the following way: $N_{\Delta_A}(\lambda) := \{$ the number of eigenvalues of A counted with multiplicities, not greater than λ, divided by $|A|\}$. We say that the integrated density of states exists in the sense of uniform convergence of the distributions (shortly: the uniform spectral convergence exists) if there is a monotone function N_{Δ_G} such that for any Følner-sequence $\{Q_n\}_{n=1}^\infty$ the functions $N_{\Delta_{Q_n}}$ uniformly converge to N_{Δ_G}. It is important to note that weak spectral convergence, that is

$$\lim_{n\to\infty} N_{\Delta_{Q_n}}(\lambda) = N_{\Delta_G}$$

for all points of continuity of N_{Δ_G} is generally much easier to prove (see the discussion in Section 7). Lenz and Stollmann proved the existence of universal spectral convergence in the case of aperiodic order for Laplacian-type operators defined by nice Delone-sets of $\mathbb{R}^d$. In our paper we study the same problem form a ring theoretical point of view. We introduce the spectral functions for the elements of continuous algebras defined by Goodearl [2] as a generalization of the original construction of John von Neumann [7]. These rings $\mathcal{R}$ are complete metric rings equipped with rank functions. We shall see in Proposition 3.1 that if $\{T_n\}_{n=1}^\infty \subset \mathcal{R}$ converge to $T \in \mathcal{R}$, then the spectral functions of T_n converge uniformly to the spectral function of T. We shall use this fact to prove uniform spectral convergence for random and deterministic operators. The examples considered in the paper are:

- Random Schrödinger operators on lattices with finite site probabilities.
- Percolation Hamiltonians.
- Pattern-invariant operators on self-similar graphs.

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2 The Goodearl-von Neumann construction

In this section we recall Goodearl's construction of rank rings [2] via Bratteli diagrams. This is a generalization of the original construction of John von Neumann [7]. Let $\mathcal{R}_1, \mathcal{R}_2, \dots$ be finite dimensional algebras over the real numbers,

$$\mathcal{R}_i := \bigoplus_{\alpha=1}^{k_i} \mathrm{Mat}_{\,n_{i,\alpha}\times n_{i,\alpha}}(\mathbb{R})\,.$$

Next we consider injective unital algebra homomorphisms $\phi_i : \mathcal{R}_i \to \mathcal{R}_{i+1}$. For each $i \geq 1$ we have k_i *nodes* $\{a^i_\alpha\}^{k_i}_{\alpha=1}$. Also, for each pair $(a^i_\alpha, a^{i+1}_\beta)$, $1 \leq \alpha \leq k_i$, $1 \leq \beta \leq k_{i+1}$ we have *weights* $w_{\alpha,\beta}$ satisfying the following compatibility condition:

- For any $1 \leq \beta \leq k_{i+1}$:
$$n_{i+1,\beta} = \sum_{\alpha=1}^{k_i} w_{\alpha,\beta} n_{i,\alpha}\,.$$
  Thus this system defines diagonal imbeddings $\phi_i : \mathcal{R}_i \to \mathcal{R}_{i+1}$.

One can associate Markov-transition probabilities
to the weight system: $M(\beta,\alpha) := \frac{w_{\alpha,\beta} n_{i,\alpha}}{n_{i+1,\beta}}$. A rank function on the inductive limit is defined as follows. Let $p_{i,\alpha} > 0$ be real numbers satisfying the relations:

$$\sum_{\alpha=1}^{k_i} p_{i,\alpha} = 1 \tag{1}$$

$$\text{For any } 1 \leq \alpha \leq k_i \quad p_{i,\alpha} = \sum_{\beta=1}^{k_{i+1}} M(\beta,\alpha) p_{i+1,\beta}\,. \tag{2}$$

Then for each $i \geq 1$ one has a normalized rank function $r_i : \mathcal{R}_i \to \mathbb{R}$, defined as

$$r_i(A_{i,1} \oplus A_{i,2} \oplus \ldots \oplus A_{i,k_i}) := \sum_{\alpha=1}^{k_i} p_{i,\alpha} \frac{\operatorname{Rank} A_{i,\alpha}}{n_{i,\alpha}} .$$

Then if $A = (A_{i,1} \oplus A_{i,2} \oplus \ldots \oplus A_{i,k_i}) \in \mathcal{R}_i$, $r_i(A) = r_{i+1}(\phi_{i+1}(A))$. Hence we defined a normalized rank function r on the inductive limit $\lim_{\overrightarrow{\phi_i}} \mathcal{R}_i$. Clearly the rank r satisfies the following conditions:

- $r(A) \geq 0$ for any $A \in \lim_{\overrightarrow{\phi_i}} \mathcal{R}_i$.
- $r(1) = 1, r(0) = 0$ and $r(A) = 0$ if and only if $A = 0$.
- $r(A + B) \leq r(A) + r(B)$.
- $r(AB) \leq r(A)$, $r(AB) \leq r(B)$.

Therefore $d(A, B) := r(A - B)$ defines a metric on the inductive limit ring. Its completion is the continuous regular ring $\mathcal{R}$ and the rank r extends to $\mathcal{R}$ satisfying the four conditions above.

3 Spectral functions

Let us consider the inductive limit system

$$\mathcal{R}_1 \xrightarrow{\phi_1} \mathcal{R}_2 \xrightarrow{\phi_2} \ldots$$

as in Section 2. For each $i \geq 1$ consider the vectorspace $\bigoplus_{\alpha=1}^{k_i} \mathbb{R}^{n_{i,\alpha}} = V_i$. The algebra $\mathcal{R}_i = \bigoplus_{\alpha=1}^{k_i} \operatorname{Mat}_{n_{i,\alpha} \times n_{i,\alpha}}(\mathbb{R})$ acts on V_i the natural way. Now we define a dimension function on the $\mathcal{R}_i$-submodules of V_i. Note that any such $\mathcal{R}_i$-submodule L is in the form $\bigoplus_{\alpha=1}^{k_i} L_{i,\alpha}$, where $L_{i,\alpha} \subseteq \mathbb{R}^{n_{i,\alpha}}$ is a linear subspace. Then

$$\dim_{\mathcal{R}_i}(\bigoplus_{\alpha=1}^{k_i} L_{i,\alpha}) := \sum_{\alpha=1}^{k_i} p_{i,\alpha} \frac{\dim_{\mathbb{R}} L_{i,\alpha}}{n_{i,\alpha}} .$$

Then of course, $\dim_{\mathcal{R}_i}(V_i) = 1$. Let $T \in \mathcal{R}_i$, then we call a $\mathcal{R}_i$-submodule $L \subseteq V_i$ a λ^--space of T if $\|T(v)\| \leq \lambda\|v\|$, for any $v \in L$, where $\|v\|$ is the Euclidean norm of v.

Similarly, we call a $\mathcal{R}_i$-submodule $N \subseteq V_i$ a λ^+-space of T if $\|T(v)\| > \lambda\|v\|$, for any $0 \neq v \in N$. Now we define $\sigma^i_T(\lambda)$ as the maximal dimension (with respect to $\dim_{\mathcal{R}_i}$!) of all the λ^--spaces of T and $\widetilde{\sigma}^i_T(\lambda)$ as the maximal dimension of all the λ^+-spaces of T.

Lemma 3.1 *For any $T \in \mathcal{R}_i$, $\sigma^i_T(\lambda) + \widetilde{\sigma}^i_T(\lambda) = 1$. If $T = \bigoplus_{\alpha=1}^{k_i} T_\alpha$ is a positive, self-adjoint element then $\sigma^i_T(\lambda)$ equals to $\sum_{\alpha=1}^{k_i} p_{i,\alpha} N_{T_\alpha}(\lambda)$, where N_{T_α} is the normalized spectral distribution function, defined in the Introduction.*

Proof. Let $S \in \mathcal{R}_i$ be the positive self-adjoint matrix such that $S^2 = T^*T$. Then $\sigma^i_S(\lambda) = \sigma^i_T(\lambda)$ and $\widetilde{\sigma}^i_S(\lambda) = \widetilde{\sigma}^i_T(\lambda)$. Indeed

$$\|T(v)\| \leq \lambda\|v\| \Leftrightarrow \langle T(v), T(v)\rangle \leq \lambda^2\|v\|^2 \Leftrightarrow \langle T^*T(v), v\rangle \leq \lambda^2\|v\|^2 \Leftrightarrow$$

$$\Leftrightarrow \langle S^2(v), v\rangle \leq \lambda^2\|v\|^2 \Leftrightarrow \|S(v)\| \leq \lambda\|v\|\,.$$

Hence in order to prove the equality $\sigma^i_T(\lambda) + \widetilde{\sigma}^i_T(\lambda) = 1$ it is enough to suppose that T is positive self-adjoint. Clearly, $\sigma^i_T(\lambda) + \widetilde{\sigma}^i_T(\lambda) \leq 1$ since the intersection of a λ^--space and a λ^+-space is always the zero vector. On the other hand, the eigenvectors of T corresponding to eigenvalues not greater than λ span a λ^--space respectively the eigenvectors corresponding to eigenvalues that greater than λ span a λ^+-space and the sum of their $\dim_{\mathcal{R}_i}$-dimensions is 1. ∎

Lemma 3.2 *For any $T \in \mathcal{R}_i$,*

$$\sigma^i_T(\lambda) = \sigma^{i+1}_{\phi_i(T)}(\lambda)\,.$$

$$\widetilde{\sigma}^i_T(\lambda) = \widetilde{\sigma}^{i+1}_{\phi_i(T)}(\lambda)\,.$$

Proof. First note that ϕ_i does not only map $\mathcal{R}_i$ into $\mathcal{R}_{i+1}$ but also it naturally associates $\mathcal{R}_{i+1}$-submodules of V_{i+1} to $\mathcal{R}_i$-submodules of V_i preserving their dimensions, that is for any $\mathcal{R}_i$-submodule L

$$\dim_{\mathcal{R}_i}(L) = \dim_{\mathcal{R}_{i+1}}(\phi_i(L))\,.$$

Simply let P_L be the orthogonal projection onto L, then we define $\phi_i(L)$ as the range of $\phi_i(P_L)$. If $L \subseteq V_i$ is a λ^--space of T, then $\phi_i(L)$ is a λ^--space of $\phi_i(T)$. Similarly, if $N \subseteq V_i$

is a λ^+-space of T, then $\phi_i(N)$ is a λ^+-space of $\phi_i(T)$. Hence the image of a maximal dimensional λ^--space must be a maximal dimensional λ^--space as well. ∎

By the previous lemma, we have spectral functions $\sigma_T(\lambda)$ and $\widetilde{\sigma}_T(\lambda)$ on the inductive limit ring $\lim_{\overrightarrow{\phi_i}} \mathcal{R}_i$.

Proposition 3.1 *Let $\mathcal{R}$ be the rank closure of $\lim_{\overrightarrow{\phi_i}} \mathcal{R}_i$ as in Section 2. Suppose that $A_i \in \mathcal{R}_i$, $A \in \mathcal{R}$ and $A_i \to A$ in the metric defined by the rank function. Then the spectral functions σ_{A_i} uniformly converge to a function σ_A and σ_A does not depend on the choice of the sequence $\{A_i\}_{i=1}^\infty$. Also, for any $A, B \in \mathcal{R}$; $\|\sigma_A - \sigma_B\|_\infty \leq d(A, B)$ where $\| \, \|_\infty$ is the L^∞-norm.*

Proof. First we need the following comparison lemma.

Lemma 3.3 *Let $T, S \in \mathcal{R}_i$ such that $r_i(T - S) = \epsilon$. Then for any $\lambda \geq 0$,*

$$|\sigma_T^i(\lambda) - \sigma_S^i(\lambda)| \leq \epsilon \text{ and } |\widetilde{\sigma}_T^i(\lambda) - \widetilde{\sigma}_S^i(\lambda)| \leq \epsilon \tag{3}$$

Proof. Let L be a λ^--space for T. Note that $\dim_{\mathcal{R}_i} \operatorname{Ker}(T - S) = 1 - \epsilon$, since $r_i(T - S) = \dim_{\mathcal{R}_i} \operatorname{Im}(T - S)$. Also,

$$\dim_{\mathcal{R}_i} \operatorname{Ker}(T - S) + \dim_{\mathcal{R}_i} L \leq 1 + \dim_{\mathcal{R}_i} (\operatorname{Ker}(T - S) \cap L) \, .$$

That is

$$\dim_{\mathcal{R}_i} (\operatorname{Ker}(T - S) \cap L) \geq \dim_{\mathcal{R}_i} L - \epsilon \, .$$

Clearly, $\operatorname{Ker}(T - S) \cap L$ is a λ^--space for S, thus our lemma follows. ∎

Now we finish the proof of our proposition. By (3), the spectral function $\sigma : \lim_{\overrightarrow{\phi_i}} \mathcal{R}_i \to L^\infty[0, \infty)$ is a Lipschitz-continuous map from our inductive limit ring to the Banach space of bounded functions. Hence σ extends to the metric completion $\mathcal{R}$ in a unique continuous way. ∎

Later we shall use the following simple version of Lemma 3.3.

Lemma 3.4 *If A, B are positive self-adjoint linear transformations in $Mat_{n \times n}(\mathbb{R})$ and $\frac{Rank_{(A-B)}}{n} \leq \epsilon$, then for any $\lambda \geq 0$: $|N_A(\lambda) - N_B(\lambda)| \leq \epsilon$.*

4 Random Schrödinger operators on lattices

Let G_d be the d-dimensional standard lattice graph. For each vertex p of G_d we have independent, identically distributed random variables $X_{(p)}$ which take the real values $J = \{c_1, c_2, \ldots, c_k\}$ with probabilities $\{p_1, p_2, \ldots, p_k\}$, $p_i > 0$. Now let $\Omega = J^{G_d}$ the associated Bernoulli state space with the product measure. Thus for each $\omega \in \Omega$, we have an operator Δ_ω:

$$\Delta_\omega f(x) = \deg f(x)\,\omega(x) - \sum_{(x,y)\in E(G_d)} f(y)\,.$$

Theorem 1 *For any Følner-sequence $\{Q_n\}_{n=1}^{\infty} \subset G_d$ the following holds: For almost all $\omega \in \Omega$, the spectral distribution functions $N_{\Delta_{Q_n}}$ converge uniformly to a function N_Δ, where $N_{\Delta_{Q_n}}$ are the normalized spectral distribution functions of the truncated Laplacians $p_{Q_n}\Delta_\omega i_{Q_n}$, where $p_{Q_n} : L^2(G_d) \to L^2(Q_n)$ is the standard projection operator and $i_{Q_n} : L^2(Q_n) \to L^2(G_d)$ is its adjoint, the imbedding operator. The function N_Δ does not depend on ω.*

Proof. First of all note that by adding a sufficiently large constant to the numbers $\{c_1, c_2, \ldots, c_k\}$ we may suppose that Δ_ω is positive operator for all $\omega \in \Omega$. The proof shall be given in several steps.

Step 1. We construct a Bratteli system as in Section 2. Consider the cubes C_i in G_d with sidelength 2^i. That is $C_i = \{0, 1, 2, \ldots, 2^i - 1\}^d$. Now let A_i be the finite set of configurations (J-valued functions) on C_i. Then $k_i = |A_i| = k^{2^{id}}$.

Step 2. Now we construct the weights $w_{\alpha,\beta}$. For each configuration $a^{i+1}_\beta \in A_{i+1}$ we associate 2^d (not necessarily different) elements of A_i. Namely, the cube C_{i+1} is partitioned into 2^d dyadic cubes of sidelength 2^i and the configuration a^{i+1}_β determines a configuration on each such subcubes. Then $w_{\alpha,\beta}$ is the number of occurences of the configuration $a^i_\alpha \in A_i$ in a^{i+1}_β.

Step 3. Now we define the parameters of our Bratelli diagrams. First of all $n_{i,\alpha} = 2^{id}$ for any $1 \leq \alpha \leq k_i$. Now let $p_{i,\alpha}$ be the probability of the configuration a^i_α. That is

$$p_{i,\alpha} := \prod_{x\in C_i} P(a^i_\alpha(x))\,,$$

where of course $P(c_i) = p_i$. Thus, $\sum_{\alpha=1}^{k_i} p_{i,\alpha} = 1$ holds.

Step 4.

Lemma 4.1 *For any* $1 \leq \alpha \leq k_i$

$$p_{i,\alpha} = \sum_{\beta=1}^{k_{i+1}} M(\beta,\alpha) p_{i+1,\beta}\,.$$

Proof. Le us consider the cube C_{i+1} with sidelength 2^{i+1} that determines 2^d dyadic subcubes of sidelength 2^i. Let us randomly choose a J-valued function on C_{i+1} (the probability of having the value c_i on the vertex x is p_i). The expected value of the number of configurations of the type a^i_α on the 2^d-subcubes is of course $2^d p_{i,\alpha}$. By the theorem of conditional expectations this value equals to $\sum_{\beta=1}^{k_{i+1}} p_{i+1,\beta} w_{\alpha,\beta}$, where $w_{\alpha,\beta}$ is the number of occurences of the configuration a^i_α in a^{i+1}_β. Since $w_{\alpha,\beta} = 2^d M(\beta,\alpha)$, our Lemma follows. ∎

Step 5. The finite dimensional algebra $\mathcal{R}_i$ is defined as $\bigoplus_{\alpha=1}^{k_i} \mathrm{Mat}_{\,2^{id}\times 2^{id}}(\mathbb{R})$, that is each element of $\mathcal{R}_i$ is represented by k_i 2^{id} by 2^{id} matrices. Each configuration a^i_α defines an element $M(a^i_\alpha) \in \mathrm{Mat}_{\,2^{id}\times 2^{id}}(\mathbb{R})$.

$$M(a^i_\alpha) v_x = \deg(x) P(a^i_\alpha(x)) v(x) - \sum_{y \in C_i \mid (x,y) \in G_d} v_y$$

where v_x is the basis vector of $\mathbb{R}^{2^{id}}$ associated to the vertex $x \in C_i$ (for the sake of simplicity we identify matrices and linear transformations). Note that $\deg(x)$ is the degree of vertex x in the subgraph spanned by C_i. Therefore we defined an element $\widetilde{\Delta}_i \in \mathcal{R}_i \subset \mathcal{R}$, where $\mathcal{R}$ is the metric completion of the inductive limit ring.

Step 6.

Lemma 4.2 *The elements* $\{\widetilde{\Delta}_i\}_{i=1}^\infty$ *form a Cauchy-sequence in* $\mathcal{R}$*, hence the limit* $\lim_{i\to\infty} \widetilde{\Delta}_i = \widetilde{\Delta}$ *is well-defined.*

Proof. We need to estimate the rank $r(\widetilde{\Delta}_i - \widetilde{\Delta}_j)$. First we partition C_j into $2^{d(j-i)}$ dyadic subcubes. Any such subcube has boundary points and non-boundary points. Boundary points are those points which are adjacent to a point of another subcube. By our definition,

$$\widetilde{\Delta}_i(v_x) = \widetilde{\Delta}_j(v_x)$$

for any non-boundary point $x \in C_j$. Hence $r(\widetilde{\Delta}_i - \widetilde{\Delta}_j)$ is bounded above by $\beta_{i,j}$, where

$$\beta_{i,j} := \frac{\{\text{the number of boundary points in } C_j.\}}{|C_j|}\,.$$

Since for large enough i, $\beta_{i,j} < \epsilon$ for any $j > i$, the lemma follows. ∎

Step 7. Now we turn to the proof of Theorem 1. Let $\{Q_n\}_{n=1}^{\infty}$ be a Følner-sequence in G_d. Let $\omega \in \Omega$ be a configuration on G_d. Pick a positive integer j and cover G_d by disjoint translates of our standard C_j-cube. That is $G_d = \bigcup_{\underline{t}}\{C_j + \underline{t}\}$, where all the coordinates of the vectors $\underline{t}$ are divisible by 2^j. We compare the following two self-adjoint transformations

$$\Delta_{Q_n} : L^2(Q_n) \to L^2(Q_n)$$

given by $\Delta_{Q_n} = p_{Q_n}\Delta_\omega i_{Q_n}$, and

$$\Delta^j_{Q_n} : L^2(Q_n) \to L^2(Q_n)$$

given by $\sum_w p_{Q_n}\Delta_\omega i_w$, where w runs through all the translated copies of C_i which are contained completely in the set Q_n.

By the Følner-property, for any $\epsilon > 0$ there exist integers j_ϵ, n_ϵ such that

- $\frac{\text{Rank}\,(\Delta_{Q_n} - \Delta^{j_\epsilon}_{Q_n})}{|Q_n|} < \epsilon\,.$

- and for any $n > n_\epsilon$, $\|s_{\widetilde{\Delta}} - s_{\widetilde{\Delta}_{j_\epsilon}}\|_\infty < \epsilon$ where $\|\ \|_\infty$ denotes the L^∞-norm.

Hence by Lemma 3.4, if $n > n_\epsilon$

$$|N_{\Delta_{Q_n}}(\lambda) - N_{\Delta^{j_\epsilon}_{Q_n}}(\lambda)| < \epsilon\,, \tag{4}$$

for any $\lambda \geq 0$.

Step 8. In order to finish the proof of our theorem it is enough to prove that for any $\epsilon > 0$

$$\limsup_{n\to\infty} \|s_{\widetilde{\Delta}_{j_\epsilon}} - N_{\Delta^{j_\epsilon}_{Q_n}}\|_\infty \leq \epsilon$$

shall hold for almost all $\omega \in \Omega$. The function N_Δ is defined as $s_{\widetilde{\Delta}}$.

Again we consider the set of all configurations A_{j_ϵ} on the cube C_{j_ϵ}. Then by Lemma 3.1:

$$s_{\widetilde{\Delta}_{j_\epsilon}}(\lambda) = \sum_{w_\alpha \in A_{j_\epsilon}} p_{j_\epsilon,\alpha} N_{w_\alpha}(\lambda)\,,$$

where N_{w_α} is the normalized spectral distribution function of the transformation Δ_{w_α}. On the other hand,

$$N_{\Delta_{Q_n}^{j_\epsilon}}(\lambda) = \sum_{w_\alpha \in A_{j_\epsilon}} R^n_{j_\epsilon,\alpha} N_{w_\alpha}(\lambda)\,,$$

where $R^n_{j_\epsilon,\alpha}$ is the number of occurences of the configuration w_α in the C_{j_ϵ}-translates contained by Q_n divided by the number of all such translates. By the Theorem of Large Numbers, for almost all $\omega \in \Omega$: $\lim_{n\to\infty} R^n_{j_\epsilon,\alpha} = P_{j_\epsilon,\alpha}$ holds for all configuration $w_\alpha \in A_{j_\epsilon}$. Hence the Theorem follows. ∎

5 Percolation Hamiltonians

In this section we apply our approximation method to prove uniform spectral convergence in the case of bond- and site-percolation Hamiltonians. The pointwise spectral convergence was proved in this case in [3] and [8].

The bond percolation Hamiltonian: Again we work on the lattice G_d. For the edges $e \in E(G_d)$ we consider independent indentically distributed random variables $X_{(e)}$, such that

$$P(X_{(e)} = 0) = p, \quad P(X_{(e)} = 1) = 1 - p\,.$$

The associated Bernoulli-state space $\Omega^B = \prod_{e\in E(G_d)}\{0,1\}$ is equipped with the standard product measure. Then each $\omega \in \Omega$ defines a subgraph $G^B_\omega \subset G_d$, where the vertex set is the lattice and $e \in E(G^B_\omega)$ if and only if $\omega(e) = 0$.

Theorem 2 *For all Følner-sequences $\{Q_n\}_{n=1}^\infty \subset G_d$ (in the original sense in the lattice) the following statement holds: For almost all $\omega \in \Omega$ the normalized spectral distribution functions $N_{\Delta_{Q_n}}$ uniformly converge to the integrated density of state function N^B_Δ. The function N^B_Δ does not depend on ω. The normalized spectral distribution functions $N_{\Delta_{Q_n}}$ are associated to the Laplacian operator of the subgraphs of G^B_ω spanned by the vertices of Q_n.*

The site percolation Hamiltonian: In the case of site percolation Hamiltonian we consider independent identically distributed random variables (with the same distribution as above)

indexed by the vertices of the lattice G_d. Again, the state space $\Omega^S = \prod_{p\in G_d}\{0,1\}$ is equipped with the product measure. For $\omega \in \Omega^S$ one has the subgraph $G^S_\omega \subset G^d$, where $e \in E(G^S_\omega)$ if and only if $\omega(p) = \omega(q) = 0$ for the endpoints of e.

Theorem 3 *For all Følner-sequences $\{Q_n\}_{n=1}^\infty \subset G_d$ the following statement holds: For almost all $\omega \in \Omega$ the normalized spectral distribution functions $N_{\Delta_{Q_n}}$ uniformly converge to the integrated density of state function N^S_Δ. The function N^S_Δ does not depend on ω. The normalized spectral distribution functions $N_{\Delta_{Q_n}}$ are associated to the Laplacian operator of the subgraphs of G^S_ω spanned by the vertices of Q_n.*

Proof. (of Theorem 2, the proof of Theorem 3 is completely analogous)

Step 1. Let A_i be the finite configuration space of all $\{0,1\}$-valued functions on lattice edges in the 2^i-cube C_i, $k_i = |A_i|$.

Step 2. For $a^i_\alpha \in A_i$ and $a^{i+1}_\beta \in A_{i+1}$ let $w_{\alpha,\beta}$ be the number of occurences of the configuration $a^i_\alpha \in A_i$ in the 2^d subconfigurations determined by $a^{i+1}_\beta \in A_{i+1}$.

Step 3. For $a^i_\alpha \in A_i$, let $p_{i,\alpha}$ the configuration probability that is

$$p_{i,\alpha} := p^{T(a^i_\alpha)}(1-p)^{S(a^i_\alpha)}\,,$$

where $T(a^i_\alpha)$ is the number of zeroes in the configuration a^i_α and $S(a^i_\alpha)$ is the number of ones in the configuration a^i_α.

Step 4.

$$p_{i,\alpha} = \sum_{\beta=1}^{k_{i+1}} M(\beta,\alpha)p_{i+1,\beta}\,.$$

follows from the conditional expectation argument as in the proof of Theorem 1.

Step 5. The finite dimensional algebra $\mathcal{R}_i$ is again isomorphic to $\bigoplus_{\alpha=1}^{k_i} \mathrm{Mat}_{\,2^{id}\times 2^{id}}(\mathbb{R})$. Each configuration a^i_α defines an element $M(a^i_\alpha) \in \mathrm{Mat}_{\,2^{id}\times 2^{id}}(\mathbb{R})$ as the associated Laplacian operator $\Delta_{a^i_\alpha} : L^2(C_i) \to L^2(C_i)$. Hence again we defined an element $\widetilde{\Delta}_i \in \mathcal{R}_i \subset \mathcal{R}$.

Step 6. The fact that the elements $\{\widetilde{\Delta}_i\}_{i=1}^\infty$ form a Cauchy sequence in $\mathcal{R}$ can be seen exactly the same way as in the proof of Theorem 1.

Step 7. Let $\{Q_n\}_{n=1}^\infty$ be a Følner-sequence in G_d. Let $\omega \in \Omega$ be a configuration on G_d. We consider the same translated copies as in the proof of Theorem 1. Let us compare the

following two self-adjoint transformations:

$$\Delta_{Q_n} : L^2(Q_n) \to L^2(Q_n)$$

where Δ_{Q_n} is the Laplacian of the subgraph of G^B_ω spanned by the vertices of Q_n and

$$\Delta^j_{Q_n} : L^2(Q_n) \to L^2(Q_n)$$

given by $\sum_w \Delta_w$, where Δ_w is the Laplacian associated to the dyadic subcube w. Again, by the Følner-property for any $\epsilon > 0$ there exist integers j_ϵ, n_ϵ such that

- $\frac{\text{Rank}\,(\Delta_{Q_n} - \Delta^{j_\epsilon}_{Q_n})}{|Q_n|} < \epsilon\,.$
- and for any $n > n_\epsilon$, $\|s_{\widetilde{\Delta}} - s_{\widetilde{\Delta}_{j_\epsilon}}\|_\infty < \epsilon\,.$

Hence by Lemma 3.4, if $n > n_\epsilon$

$$|N_{\Delta_{Q_n}}(\lambda) - N_{\Delta^{j_\epsilon}_{Q_n}}(\lambda)| < \epsilon\,, \tag{5}$$

for any $\lambda \geq 0$.

<u>Step 8.</u> The final argument, that for any $\epsilon > 0$

$$\limsup_{n\to\infty} \|s_{\widetilde{\Delta}_{j_\epsilon}} - N_{\Delta^{j_\epsilon}_{Q_n}}\|_\infty \leq \epsilon$$

holds for almost all $\omega \in \Omega$ follows exactly the same way as in the case of the random Schrödinger operators. ∎

6 Pattern-invariant operators on self-similar graphs

Let $G(V, E)$ be a connected infinite graph with bounded vertex degrees. A pattern-invariant operator $\mathcal{A} : L^2(V) \to L^2(V)$ (given by its operator kernel $A : V \times V \to \mathbb{R}$) satisfies the following properties:

- $A(x, y) = 0$ if $d_G(x, y) > r_{\mathcal{A}}$, where d_G is the shortest path distance on the vertices. The value $r_{\mathcal{A}}$, the propagation of $\mathcal{A}$, depends only on $\mathcal{A}$.

- $A(x,y) = A(\psi(x),\psi(y))$ if ψ is a graph isomorphism (there can be more than one) between the $r_{\mathcal{A}}$-ball around x and the $r_{\mathcal{A}}$-ball around $\psi(x)$, mapping of course x to $\psi(x)$.

Note that

$$\mathcal{A}(f)(x) = \sum_{y \in V} A(x,y) f(y)\,,$$

for $f \in L^2(V)$. These sort of operators were considered in [4],[5] and [8].

Now we define self-similar graphs (there are plenty of definitions we choose one which fits our purposes). First we fix two positive integers d and k. Let G_1 be a finite connected graph with vertex degree bound d with a distinguished subset $S_1 \subset V_1(G_1)$, which we call the set of connecting vertices. Now we consider the graph $\widetilde{G}_1$, which consists of k disjoint copies of G_1 with following additional properties:

- The graph G_1 is identified with the first copy.
- In each copy the vertices associated to a connecting vertex of G_1 is a connecting vertex of the graph $\widetilde{G}_1$.

The graph G_2 is defined by adding some edges to $\widetilde{G}_1$ such that both endpoints of these new edges are connecting vertices. The resulting graph should still have vertex degree bound d. Finally the subset $S_2 \subset V(G_2)$ is chosen as a subset of the connecting vertices of $\widetilde{G}_1$ such that $S_2 \cap V(G_1) = \emptyset$. That is $G_1 \subset G_2$ is a subgraph and vertices of G_2 which are not in G_1 are connecting vertices. Inductively, suppose that the finite graphs $G_1 \subset G_2 \subset \ldots \subset G_n$ are already defined and the vertex degrees in G_n are not greater than d. Also suppose that a set $S_n \subset V(G_n)$ is given and $S_n \cap V(G_{n-1}) = \emptyset$. Now the graph $\widetilde{G}_n$ consists of k disjoint copies of G_n and

- The graph G_n is identified with the first copy.
- In each copy the vertices associated to a connecting vertex of G_n is a connecting vertex of the graph $\widetilde{G}_n$.

Again, G_{n+1} is constructed by adding edges to $\widetilde{G}_n$ with endpoints which are connecting vertices, preserving the vertex degree bound condition. The set of connecting vertices S_{n+1}

is chosen as a subset of the connecting vertices of $\widetilde{G}_n$, such that $S_{n+1} \cap V(G_n) = \emptyset$. The union of the graphs $\{G_n\}_{n=1}^{\infty}$ is connected infinite graph with vertex degrees not greater than d. The graph G is *self-similar* if $\lim_{n\to\infty} \frac{|S_n|}{|V(G_n)|} = 0$. In this case $\{V(G_n)\}_{n=1}^{\infty}$ form a Følner-sequence of G. Note that all Euclidean lattices is self-similar in our sense.

Theorem 4 *Let G be a self-similar graph and $\mathcal{A}$ be a self-adjoint pattern- invariant operator. Then there exists a real function $N_{\mathcal{A}}$ such that for any Følner-sequence $\{Q_n\}_{n=1}^{\infty} \subset V(G)$, $N_{\mathcal{A}_{Q_n}}$ uniformly converge to $N_{\mathcal{A}}$, where $N_{\mathcal{A}_{Q_n}}$ are the spectral distribution functions of the truncated operators $p_{Q_n}\mathcal{A}\, i_{Q_n}$.*

In this case we have the simplest possible Bratteli system (the one originally considered by John von Neumann):

$$\mathrm{Mat}_{\,l\times l}(\mathbb{R}) \xrightarrow{\phi_1} \mathrm{Mat}_{\,kl\times kl}(\mathbb{R}) \xrightarrow{\phi_2} \dots$$

That is $\mathcal{R}_i = \mathrm{Mat}_{\,k^{i-1}l\times k^{i-1}l}(\mathbb{R})$, where $l = |V(G_1)|$. For each $i \geq 1$, $k_i = 1$ and each weight equals to k. Then $\widetilde{\Delta}_i \in \mathcal{R}_i$ is just the operator $\mathcal{A}_{G_i}$. The proof of Theorem 4 is completely analogous to the one of Theorem 1 and left for the interested reader.

7 A general conjecture

One can easily see that for our self-similar graphs, uniform spectral convergence holds even for the random Schrödinger operators and percolation Hamiltonians. For nice Markovian substitution systems our method works without any trouble. The most general graphs for which universal spectral convergence might hold are the following ones.

Let $G(V, E)$ be a connected infinite graph with bounded vertex degrees. We suppose that G is an amenable graph, that is G has Følner-sequence $\{F_n\}_{n=1}^{\infty}$. The r-pattern of a vertex $x \in V$ is the graph automorphism class of the rooted ball around x. That is $x, y \in V$ have the same r-pattern if there exists a graph isomorphism ϕ between the balls $B_r(x)$ and $B_r(y)$ such that $\phi(x) = y$. Denote by $P_r(G)$ the finite set of all possible r-patterns in G. We say that G is an *abstract quasicrystal graph* if for any $\alpha \in P_r(G)$ there exists a *frequency* $P(\alpha)$ such that

- For *any* Følner-sequence $\{Q_n\}_{n=1}^{\infty}$:

$$\lim_{n\to\infty} \frac{|Q_n^\alpha|}{|Q_n|} = P(\alpha)\,,$$

where $Q_n^\alpha \subseteq Q_n$ is the set of vertices in Q_n having the r-pattern α.

The graphs considered by Lenz and Stollmann are such abstract quasicrystal graphs. Another interesting examples are the Cayley graphs of finitely generated amenable groups.

Conjecture 1 *If G is an abstract quasicrystal graph, then for the random Schrödinger operators, percolation Laplacians as well as self-adjoint pattern invariant operators associated to G the uniform spectral convergence exists.*

Note that for the Laplacian operator the existence of the integrated density of states in the weak sense of convergence follows from the argument of Theorem 3. [1]. Even for the general pattern-invariant operators and the random operators the limit graph method yields the existence of the integrated density of states. The problem is to handle the jumps of the integrated density of states. One should prove that for each value λ, where the integrated density of state function jumps the normalized dimensions of the eigenfunctions of the truncated operators on the Følner-sequence converge exactly to the size of the jump. The same problem emerges in the theory of L^2-invariants in the so-called Approximation Theorem (see [6]).

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